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NUMERICAL SOLUTION OF A NONSTEADY DIFFERENTIAL EQUATION OF HEAT CONDUCTION

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The use of a "floating" weight is suggested in the numerical solution of a parabolic differential equation of heat conduction with variable coefficients in integral-mean temperatures, used in the calculation of thermal expansions of turbine components. Recommendations are given for the determination of the optimum weights.

Heat-conduction problems which are reducible to one-dimensional problems, particularly in the calculation of the distribution of the integral-mean temperatures of turbine components for the determination of their thermal expansions [1, 2], lead to the following system of differential equations:

$$\frac{1}{a}\cdot\frac{\partial\vartheta}{\partial\tau} = L\vartheta + G(z, \tau),\ \vartheta = \vartheta(z, \tau),\ 0 \leqslant z \leqslant H,\ 0 \leqslant \tau \leqslant T; \tag{1}$$

$$\frac{\partial\vartheta}{\partial z} = \nu_0(\tau)\,\vartheta - \mu_0(\tau)\Big|_{z=0},\ \frac{\partial\vartheta}{\partial z} = -\nu_n(\tau)\vartheta + \mu_n(\tau)\Big|_{z=H}; \tag{2}$$

$$\vartheta|_{\tau=0} = \varphi_0(z), \tag{3}$$

where $L\vartheta = \partial^2\vartheta/\partial z^2 + A(z, \tau)\partial\vartheta/\partial z - B(z, \tau)\vartheta$, A, B, G, ν, and μ are assigned functions (B and $\nu > 0$); a is the coefficient of thermal diffusivity.

The system (1)-(3) will be solved numerically on the grid

$$\bar{\omega}_{h\Delta\tau} = \bar{\omega}_h \times \bar{\omega}_{\Delta\tau} = \{(ih,\ j\Delta\tau),\ i = 0, 1, 2, \ldots, n,\ j = 0, 1, 2, \ldots, j_m\} \tag{4}$$

with steps $h = H/n$ and $\Delta\tau = T/j_m$.

Designating the value of the unknown grid function at the node $(z_{i,j})$ as $\theta_{i,j}$ and introducing the required number η of real parameters, also grid functions in the general case, we obtain a parametric family of difference schemes approximating the system (1)-(3).

The approximation of Eq. (1) on a six-point pattern is written as

$$\frac{1}{a}\theta_\tau = \Lambda^*\theta_\eta + \tilde{G}|_i,\ 0 < i < n,\ 0 \leqslant j \leqslant j_m, \tag{5}$$

where $\theta_\tau = (\theta_{j+1} - \theta_j)\Delta\tau$, $\Lambda^* = \Lambda + \tilde{A}l - \tilde{B}$, $\Lambda\theta_i = (\theta_{i+1} - 2\theta_i + \theta_{i-1})/h^2$, $l\theta_i = (\theta_{i+1} - \theta_{i-1})/2h$ are linear operators while $\theta_\eta = \eta\theta_{j+1} + (1 - \eta)\theta_j$.

The coefficients of Eq. (1) are determined for each time layer with its weight

$$\tilde{X} = \eta_X X_{j+1} + (1 - \eta_X) X_j,\ X = A,\ B,\ G. \tag{6}$$

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The boundary conditions (2) are approximated by a conservative scheme, using the following procedure. We multiply Eq. (1) by a function $F(z, \tau)$ such that $FA = \partial F/\partial z$, i.e.,

$$F = \exp(\int A dz), \tag{7}$$

and we integrate it in the limits z_0, z_1. We obtain

$$L^*\vartheta = \left[F\frac{\partial\vartheta}{\partial z}\right]_{z_0}^{z_1} - \int_{z_0}^{z_1} FB\vartheta dz + \int_{z_0}^{z_1} FGdz - \frac{1}{a}\int_{z_0}^{z_1}\frac{\partial\vartheta}{\partial\tau}Fdz = 0. \tag{8}$$

It is easy to show that the operator $L^*\vartheta$; with an accuracy $O(\delta^2)$, $\delta = z_1 - z_0$, over a small enough interval $[z_0, z_1]$, is equivalent to the equation

$$L^*\vartheta + O(\delta^2) = \left[F\frac{\partial\vartheta}{\partial z}\right]_{z_0}^{z_1} - \vartheta\Big|_\xi\int_{z_0}^{z_1} FBdz + \int_{z_0}^{z_1} FGdz - \frac{1}{a}\cdot\frac{\partial\vartheta}{\partial\tau}\Big|_\xi\int_{z_0}^{z_1} Fdz, \tag{9}$$

where $\xi = z_0$ or z_1, since

$$\int_{x_0}^{x} uvdx - u|_{x_0}\int_{x_0}^{x} vdx = u'v|_{x_0}\int_{x_0}^{x}(x - x_0)\,dx = O(\delta^2),$$

which follows from a Taylor series expansion of u and v in the vicinity of x_0, is satisfied for the analytical functions $u(x)$ and $v(x)$ in the vicinity $\delta = x - x_0$ of the point x_0.

Superposing the boundary conditions (2) and Eq. (9) and setting $\xi = z_0$ for the point $z = 0$ $(i = 0)$ and $\xi = z_1$ for $z = H$ $(i = n)$, we obtain, for $i = 0$, for example,

$$\frac{1}{a}\cdot\frac{\partial\vartheta}{\partial\tau}\Big|_{z=0}\int_{z_0}^{z_1} Fdz = \left[F\frac{\partial\vartheta}{\partial z}\right]_{z_1} - F\nu_0\vartheta|_{z_0} + F|_{z_0}\mu_0 - \vartheta|_{z_0}\int_{z_0}^{z_1} FBdz + \int_{z_0}^{z_1} FGdz + O(\delta^2).$$

The approximation of the latter expression and of a similar one for the end of the rod $(i = n)$ on a four-point pattern is written

$$\frac{1}{a}\theta_\tau = P_i\theta_\eta + \tilde{D}_G|_i, \quad i = 0,\ n;\ 0 \leqslant j \leqslant j_m, \tag{10}$$

where the operator $P_0 = C_0 l - \tilde{D}_0$, $P_n = -C_n l - \tilde{D}_n$, while the coefficients $\tilde{D}$ and $\tilde{D}_G$ are determined from (6) if one sets $X = D$ and D_G.

In turn,

$$C_0 = \frac{F|_{z_1}}{V},\quad C_n = \frac{F|_{z_0}}{V},\quad V = \int_{z_0}^{z_1} Fdz,\quad F_0 = F|_{z=0},\quad F_n = F|_{z=H}, \tag{11}$$

$$D_{G_i} = \frac{1}{V}\left(F_i\mu_i + \int_{z_0}^{z_1} FGdz\right),\quad D_i = \frac{1}{V}\left(F_i\nu_i + \int_{z_0}^{z_1} FBdz\right),\quad i = 0,\ n,$$

where $z_0 = 0$ and $z_1 = h/2$ are taken for the point $i = 0$ while $z_0 = H - h/2$ and $z_1 = H$ are taken for $i = n$.

The initial condition is approximated exactly:

$$\theta_{j=0} = \varphi_0|_i. \tag{12}$$

We note that the difference approximation (10) of the boundary conditions (2) of the third kind is applicable for writing difference expressions for boundary conditions of the first and second kinds. In fact, if $\mu = \nu\vartheta_f$, where ϑ_f is the temperature of the medium, is valid for the third boundary problem, then for $\nu \to \infty$ we obtain an approximation of the first boundary problem (the Dirichlet problem), in which case ϑ_f are assigned values of the unknown function at the boundary. Taking μ and ν as independent and setting $\nu = 0$, we have an approximation of the secondary boundary problem (the Neumann problem).

Let us consider the possible interpretation of the function F. In a one-dimensional differential equation of heat conduction [2], which is valid for a hollow cone with a variable wall thickness and particularly for a cylinder and a disk, the coefficient

$$A(z) = \frac{1}{F_n} \cdot \frac{dF_n}{dz},$$

where F_n is the cross-sectional area. Then, in accordance with (7), $F = \exp(\int dF_n/F_n) = F_n$. Finally, for a disk of constant cross section (z = r) we have A(z) = 1/r and F = r, while for a cylinder of constant cross section A(z) = 0 and F = 1.

The difference approximation (5), (10), (12) of the system (1)-(3) comes down to the following system of linear equations with a three-diagonal determinant:

$$\theta_{i,j} = \varphi_i \theta_{k,j+1} + \Phi_i, \quad i = 0,\ n,$$
$$\theta_{i,j} = p_i \theta_{i-1,j+1} + q_i \theta_{i+1,j+1} + \Phi_i, \quad 0 < i < n, \tag{13}$$

where for points 0 < i < n

$$p_i = f_i(1 - a_i)\eta, \quad q_i = f_i(1 + a_i)\eta, \quad f_i = 1/[(2 + b_i)\eta + 1/\mathrm{Fo}],$$
$$\Phi_i = f_i\{(1 - a_i)(1 - \eta)\theta_{i-1,j} + [1/\mathrm{Fo} - (1 - \eta)(2 + b_i)]\theta_{i,j} + (1 + a_i)(1 - \eta)\theta_{i+1,j} + g_i\}, \tag{14}$$
$$a_i = \tilde{A}_i h/2, \quad b_i = \tilde{B}_i h^2, \quad g_i = \tilde{G}_i h^2, \quad \mathrm{Fo} = a\Delta\tau/h^2,$$

while for the boundary points i = 0 and n

$$\varphi_i = f_i c_i \eta, \quad f_i = 1/[(c_i + d_i)\eta + 1/\mathrm{Fo}],$$
$$\Phi_i = f_i\{[1/\mathrm{Fo} - (1 - \eta)(c_i + d_i)]\theta_{i,j} + c_i\theta_{k,j} + d_{Gi}\}, \tag{15}$$
$$d_i = \tilde{D}_i h^2, \quad d_{Gi} = \tilde{D}_{Gi} h^2, \quad c_i = C_i h.$$

In (13) and (15) k = 1 if i = 0 and k = n − 1 if i = n. The system (13), connecting the two temperature–time layers j and j + 1, is easily solved by trial run. The solution is sought in the form

$$\theta_{i,j+1} = \psi_i \theta_{i-1,j+1} + \Psi_i. \tag{16}$$

The trial-run coefficients are

$$\psi_i = p_i/(1 - q_i\psi_{i+1}), \quad \Psi_i = (\Phi_i + q_i\Psi_{i+1})/(1 - q_i\psi_{i+1}).$$

Since $\psi_n = \varphi_n$ and $\Psi_n = \Phi_n$, all the coefficients up to ψ_1 and Ψ_1, inclusively, the temperature at the point

$$\theta_{0,j+1} = (\Phi_0 + \Psi_1\varphi_0)/(1 - \psi_1\varphi_0),$$

and finally, in accordance with (16), the entire temperature–time layer j + 1 are calculated.

Let us study the convergence of the proposed difference scheme.

The error of the scheme is written as $y = \theta - \vartheta$, omitting the indices [3, 5].

The system of equations determining the error y is obtained by superposing the expressions for the errors with Eqs. (5), (10), and (12).

Allowing for the linearity of the operators, we write

$$\frac{1}{a} y_\tau = \Lambda^* y_\eta + \psi, \quad (z, \tau) \in \omega_{h\Delta\tau},$$
$$\frac{1}{a} y_\tau = P y_\eta + \psi_\Gamma, \quad i = 0,\ n, \quad \tau \in \bar{\omega}_{\Delta\tau}, \tag{17}$$
$$y|_{\tau=0} = 0, \quad z \in \bar{\omega}_h,$$

where

$$\psi = \Lambda^*\vartheta_\eta - \frac{1}{a}\vartheta_\tau + \tilde{G}, \quad \psi_\Gamma = P\vartheta_\eta - \frac{1}{a}\vartheta_\tau + \tilde{D}_G$$

are the errors in the approximation of the differential equation and the boundary conditions, respectively. We denote the time derivative by a dot ($\dot{\vartheta}$) and the quantities at the time j + 0.5 by an upper bar ($\bar{\vartheta}$).

After transformations, the error in the approximation of the equation, with allowance for $L = \Lambda^* + O(h^2)$, can be represented in the form

$$\psi = L\bar{\vartheta} + \bar{G} - \frac{1}{a}\bar{\dot{\vartheta}} + \left[(\eta - 0.5)L\dot{\bar{\vartheta}} + (\eta_A - 0.5)\frac{\partial\bar{\vartheta}}{\partial z}\dot{\bar{A}} - (\eta_B - 0.5)\bar{\vartheta}\dot{\bar{B}} + (\eta_G - 0.5)\dot{\bar{G}}\right]\Delta\tau + O(h^2 + \Delta\tau^2). \tag{18}$$

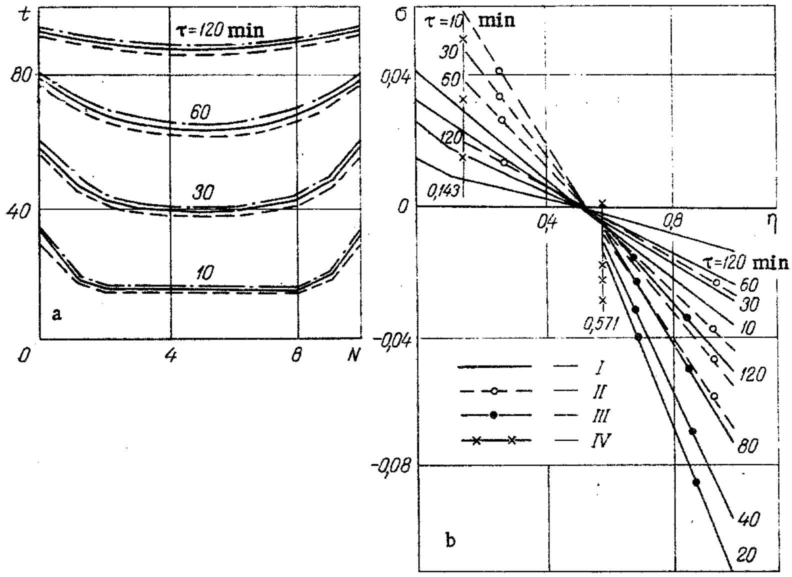

Fig. 1. Effect of the time step on the relative error in the weight function for the unknown function: a) temperature variation along length of rod [initial distribution t_0 = 0°C, temperature of medium t_f = 100°C (Bi = Bi_0 = Bi_n = 0.67, Fo = $a\Delta\tau/h^2$ = 0.25; solid curve: exact solution; dash−dot curve: η = 0; dashed curve: η = 1]; b) relative error for the fifth point: I) Fo = 0.25, II) 0.5, III) 1, IV) limit of positive approximation; N) number of point.

The error in the approximation of the boundary conditions, with allowance for the fact that the first derivative is approximated symmetrically relative to cross sections at a distance h/2 from the ends, is written as

$$\psi_\Gamma = \frac{1}{V} L^* \bar{\vartheta} + [(\eta - 0.5) P\bar{\vartheta} + (\eta_{DG} - 0.5)\bar{D}_G - (\eta_D - 0.5)\bar{\vartheta}\bar{D}]\Delta\tau + O(h^2 + \Delta\tau^2). \tag{19}$$

If, with allowance for (1) and (8), we set

$$\eta_Y = 0.5 \quad \text{or} \quad \eta_Y = 0.5 + K\frac{h^2}{\Delta\tau}, \tag{20}$$

where Y = A, B, G, P, and D_G and the index is absent, then when Fo is finite and K is a constant independent of h and $\Delta\tau$ the approximation errors (19) and (20) are

$$\psi = O(h^2 + \Delta\tau^2). \tag{21}$$

For η_Y = 0 or 1 the error is written

$$\psi = O(h^2 + \Delta\tau).$$

It is now seen that the procedure used in the approximation of the boundary conditions allows one to introduce a time variable into the difference expression for the latter, to use a weight η for the unknown function, and to obtain the same order of approximation at all points of the space−time grid, including the boundary points.

Samarskii [3] suggests a scheme for approximating the boundary conditions which also permits one to obtain the estimate (21) constructed on a Taylor series expansion. For cylinders with coefficients B and G which are constant over z at a length h/2 from the ends or for an infinite plate with a distributed heat source in the interval (h/2, H − h/2) the difference expressions obtained by the two methods coincide. For a solid disk, however, the differential equation has a singularity of the type 1/r for r = 0 and in such cases, as well as when B and G are variable over z, one must give preference to the integral approach to the construction of the approximating expressions.

Stability of the solution is obtained by applying only approximating expressions of the positive type, which assure monotonic convergence satisfying the maximum principle [3, 4, 5]. The conditions determining the positive approximation of the system (5), (10), (12), with allowance for Eqs. (14) and (15), can be written as

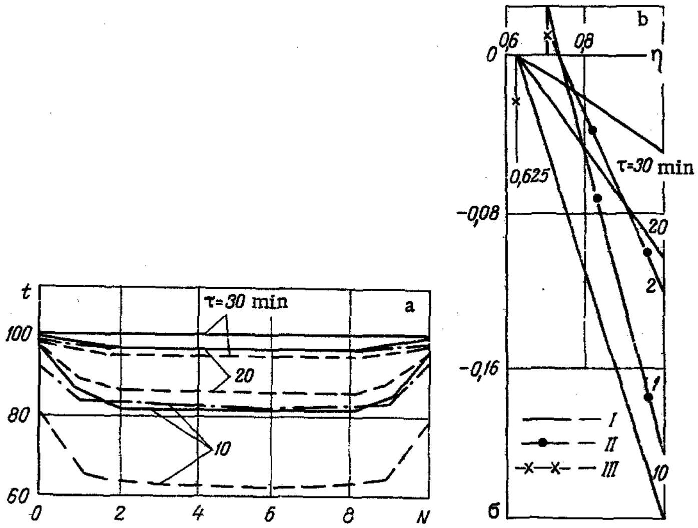

Fig. 2. Effect of the coefficient of heat transfer on the relative error in the weight function for the unknown function: a) temperature variation along length of rod, initial distribution t_0 = 0°C, temperature of medium t_f = 100°C; solid curve) exact solution; dash–dot curve) η = 0.625; dashed curve) η = 1; Bi = Bi_0 = Bi_n = 6.67, Fo = 0.5; b) relative error for fifth point [1) Bi = 6.67; II) Bi = 133, Fo = 0.05, III) limit of positive approximation]; N) number of point.

$$|a_i| \leqslant 1, \quad 0 \leqslant \eta \leqslant 1, \quad 1/\mathrm{Fo} - (1-\eta)(c_k + B_k) \geqslant 0, \tag{22}$$

where c_k = 2, c_0, c_n, B_k = b_i, d_0, d_n, which imposes the following limits on the step in z:

$$h \leqslant 2/\sup_{ij}|A|, \quad 0 < i < n. \tag{23}$$

It is easy to show that the fulfillment of (22) and (23) assures the stability of the trial-run algorithm, since the following are valid in this case:

$$0 \leqslant \varphi_i < 1, \quad i = 0,\ n, \quad p_i > 0, \quad q_i > 0, \quad p_i + q_i < 1, \quad 0 < i < n. \tag{24}$$

The last equation of (22), which connects the characteristics of the space–time grid (Fo = $a\Delta\tau/h^2$) and the weight η of the unknown function, is traditionally solved relative to the time step

$$\Delta\tau \leqslant h^2/[a(1-\eta)(c_k + B_k)], \tag{25}$$

the weight η is fixed for the grid $\bar{\omega}_{h\Delta\tau}$, and various approximation schemes are analyzed depending on the numerical value adopted for η. For example, explicit (η = 0), purely implicit (η = 1), the Crank–Nicholson scheme (η = 1/2), and others, each having its own characteristics of convergence and stability. For the explicit scheme the expression (25) limits the step in time [1, 6]. The purely implicit scheme has no limits on the step $\Delta\tau$, but its rate of convergence, as in the preceding explicit scheme, equals $O(h^2 + \Delta\tau)$. The Crank–Nicholson scheme [$O(h^2 + \Delta\tau^2)$], like other schemes with $\eta \geq 1/2$, does not require that the relation (25) be satisfied in some cases [6, 7], but then it does not provide monotonic convergence.

The present report proposes to reject the fixing of the weight η on the grid $\bar{\omega}_{h\Delta\tau}$ and to consider the weight η of the unknown function as a function on the space–time grid (4) determined at each point (i, j) by the relation ("floating weight")

$$\eta \leqslant 1 - 1/[\mathrm{Fo}(c_k + B_k)], \tag{26}$$

where c_k = 2, B_k = b_i, 0 < i < n, c_k = c_i, B_k = d_i, and i = 0, n. In this case stability is assured by an approximation of the positive type, and with the proven approximation (20), (21), according to the Lax theorem [9], the solution of the difference problem converges monotonically to the solution of the system (1)–(3) with a rate $O(h^2 + \Delta\tau^2)$.

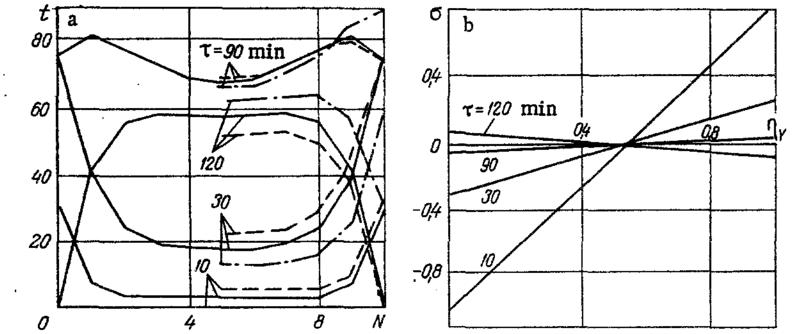

Fig. 3. Relative error in the weight function for the coefficients of the equations: a) temperature variation along length of rod, initial distribution; $t_0 = 0°C$, temperature of medium $t_f = 3.33\tau - 0.0278\tau^2 °C$; (Bi = 0.67; $Bi_0 = Bi_n = 6700$; Fo = 0.5); solid curve) exact solution; dash–dot curve) $\eta_Y = 0$; dashed curve) $\eta_Y = 1$; b) relative error σ for fifth point.

In fact, the error y in the solution of the problem (1)-(3), determined by the stable system (17) since it differs from (5), (10), (12) only on the right side, depends uniformly and continuously with respect to h and $\Delta\tau$ on ψ and ψ_T and also approaches zero as h and $\Delta\tau$ approach zero. In actual calculations, however, the steps h and $\Delta\tau$ are limited below mainly by the acceptability of the time grid.

In the approximation of the system (1)-(3) the weights (the numerical parameters $0 < \eta < 1$) introduced for the unknown function and for the variable coefficients have the optimum values within the interval (0, 1) in the sense of the minimum error in the solution, since it follows from Eqs. (18) and (19) that the cofactors to $\Delta\tau$ are opposite in sign for $\eta_Y = 0$ and $\eta_Y = 1$. Recommendations on the choice of the optimum weights can be obtained from numerical experiments comparing the difference solution with the exact solution. Moreover, an a priori examination of the conditions of convergence shows that the weight η of the temperature function determines both the stability and the accuracy of the difference solution. The lower limit to the values of η is stipulated by an approximation of the positive type which assures stability. The weights η_Y for the coefficients of the system (1)-(3) affect only the accuracy of the solution. Consequently, it is natural to conduct the experiment in two stages. First, having eliminated the effect of the variability of the coefficients of the system with time, one studies the optimum values of the weight of the temperature function (we note that in the numerical calculation the coefficients of the system are constant with respect to τ in each calculating step in time). In the second stage one must consider the effect of the weights of the coefficients of the system on the accuracy of the solution, using the recommendations on the determination of the optimum weight of the temperature function obtained in the first stage. The presence of two groups of numerical parameters allows us a certain arbitrariness in the choice of their optimum values.

The results of calculations of the difference and exact [10] solutions of a heat-conduction problem programmed on a computer were compared in the numerical experiment. A solid bounded cylinder (R = 0.2 m, length L = 1 m), divided by the grid (4) into 11 sections (n = 10) and with constant physical characteristics λ = 35 W/mg and $a = 0.83 \cdot 10^{-5}$ m²/sec, was taken as the region of determination of the temperature function. In this case for Eq.(1) we have

$$A = 0, \quad B = \mathrm{Bi}/(2R^2), \quad G = Bt_f, \quad \mathrm{Bi} = \alpha R/\lambda,$$

while the boundary conditions are written as

$$-\frac{\partial\vartheta}{\partial n} = \frac{\mathrm{Bi}}{R}(\vartheta - t_f),$$

where α and t_f are the coefficient of heat exchange and the temperature at the side (Bi) and end (Bi_0, Bi_n) surfaces. To determine the optimum weight η of the unknown function we studied the behavior of the relative error

$$\sigma = (\theta - \vartheta)/\vartheta$$

in the functions Bi and Fo, for uniform and nonuniform initial distributions, and with symmetric and asymmetric boundary conditions at the middle (fifth) point of the rod. Examples of the calculations with a constant initial distribution and constant boundary conditions are presented in Figs. 1 and 2. In the second part of the calculations the relative error was analyzed as a function of η_G (Fig. 3), since an exact solution was obtained only for constant, although asymmetric, coefficients of heat exchange, and the variability of G was determined by the time variation of the temperature of the medium.

In generalizing the results of the numerical experiment, one can propose the following scheme for choosing the weight of the unknown function.

The weight for the unknown temperature function must be taken as equal to 0.5, if this is allowed by Eq. (26), or as the least of the weights providing for a positive approximation.

For the heat-conduction equation in the ranges of the coefficients determined by the boundary conditions occurring in steam turbines one can take the weights η_Y, Y = B, G, D, D_G, as equal to 0.5.

However, these recommendations are not definitive only for a problem of more general form with the coefficients of the system (1)-(3). In considering a heat-conduction problem with the conversion of boundary conditions of the third kind into those of the first kind as a limiting transition as the coefficient of heat transfer approaches infinity, one can note that as the latter increases it is natural to take a weight ever closer to unity for the coefficient which allows for the temperature of the surrounding medium. In the limit, i.e., for boundary conditions of the first kind, a weight of unity gives a zero error with respect to time. It may turn out that for the coefficients the weight must be chosen from relations like (26).

Thus, the possibility of controlling the behavior of the solution using weights in calculations on grids with relatively large steps is demonstrated. The investigation of the choice of the optimum weights for the coefficients of the system (1)-(3) must be extended in this direction.

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